ISRAEL JOURNAL OF MATHEMATICS, Vol. 33, No. 1, 1979

A NEW PROOF OF THE MAUREY–PISIER THEOREM

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ABSTRACT
In this paper we give a new proof of the Maurey–Pisier theorem about the finite representation of l_{p_E} and l_{q_E} in any infinite dimensional Banach space E.

1. Introduction

The key theorem in the now fastly growing theory of type and cotype of Banach spaces is the theorem of Maurey and Pisier [4], stating that if p_E is the supremum of all types of an infinite dimensional Banach space E, and q_E is the infinum of all cotypes, then l_{p_E} and l_{q_E} are both finitely representable in E. This result establishes a link between probabilistic properties and geometric properties of Banach spaces, and plays an important role in the modern study of the geometry of Banach spaces.

However, the original proof of the theorem given in [4] is rather complicated. We shall present in this paper a shorter, and, to our belief, more natural proof of this theorem.

We assume that the reader is familiar with the basic theory of type and cotype. The relevant notations will be introduced in the next section, but let us first review briefly our proof.

We start as do Maurey and Pisier in their proof [4] with a finite sequence in E, almost attaining equality in the inequality in the definition of type (or cotype). This sequence is manipulated, using some lemmas taken from [4] (lemmas 2.1–2.3 in our paper), but then applying a new geometry-oriented argument, to obtain from it a new sequence which "behaves" better, in the sense that it satisfies formula (2) (resp. (4)) below in the case of cotype (resp. type). Further

Received October 29, 1978 and in revised form May 10, 1979

manipulation of this sequence, using the combinatorial theorems of Brunel and Sucheston [1] about unconditional and spreading-invariant subsequences in Banach spaces, and then the deep theorem of Krivine [3] about finite representation of l_p spaces in Banach spaces, completes the proof by producing yet another sequence spanning a finite l_p subspace of E as desired. This final construction is relatively simple in the case of cotype, but presents some technical difficulties in the case of type, as the required inequalities all go in the "wrong" direction. These difficulties are resolved in the final lemma 2.9.

The ideas in this proof have already been used by the authors in [5] to prove a similar theorem for somewhat different notions of type and cotype.

2. Proof of the Maurey–Pisier theorem

Let us first introduce several definitions related to the notions of type and cotype.

Recall that a Banach space E is said to be of type $1 \leqq p \leqq 2$ (resp. of cotype $q \geqq 2$) if there exists a constant $C > 0$ such that for each finite sequence $x_1, \cdots, x_n \in E$, we have

$$\left(\int \left\| \sum_{i=1}^{n} x_i \varepsilon_i(t) \right\|^p dt\right)^{1/p} \leqq C\left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p}$$

$$\left(\text{resp.} \quad \left(\sum_{i=1}^{n} \|x_i\|^q\right)^{1/q} \leqq C\left(\int \left\| \sum_{i=1}^{n} x_i \varepsilon_i(t) \right\|^q dt\right)^{1/q}\right)$$

where $\varepsilon_i(t)$ are the Rademacher variables. The smallest such C is denoted by C_p (resp. C_q) and is called the p-type (resp. q-cotype) constant of E. By a result of Kahane [2], all $L^p(E)$-norms are equivalent on the linear span of elements of the form $\varepsilon_j(t)x_j$, $j \geqq 1$, $x_j \in E$, and so, up to changing the constant C, we may replace the exponent in the integrals by any other exponent in $[1, \infty)$.

Let

$$p_E = \sup\{p : E \text{ is of type } p\},$$

$$q_E = \inf\{q : E \text{ is of cotype } q\}.$$

(E need not be of type p_E nor of cotype q_E.)

Let the infinite dimensional Banach space E be given and fixed (unless otherwise stated) henceforth. For each $1 \leqq p \leqq 2$ and each integer n, let $\psi_p(n)$ be the smallest positive constant ψ such that, for every n elements $x_1, \cdots, x_n$ of E,

$$\left(\int \left\| \sum_{i=1}^{n} x_i \varepsilon_i(t) \right\|^p dt\right)^{1/p} \leqq \psi\left(\sum_{i=1}^{n} \|x_i\|^p\right)^{1/p}.$$

Similarly, for each $2 \leqq q$ and each integer n, we define $\varphi_q(n)$ as the smallest positive constant φ such that, for every n elements $x_1, \cdots, x_n$ of E,

$$\left(\sum_{i=1}^{n} \|x_i\|^q\right)^{1/q} \leqq \varphi\left(\int \left\| \sum_{i=1}^{n} x_i \varepsilon_i(t) \right\|^q dt\right)^{1/q}$$

(see [4]). Evidently, the sequences $\{\psi_p(n)\}_{n\geqq 1}$, $\{\varphi_q(n)\}_{n\geqq 1}$ are increasing.

The following preliminary lemmas (2.1–2.3) are taken from Maurey and Pisier [4], and are cited here for the sake of completeness. (The first lemma is by now well-known and is used extensively in Banach-space theory.)

LEMMA 2.1. *For each $1 \leqq p \leqq 2$ and $q \geqq 2$, the sequences $\{\varphi_q(n)\}_{n\geqq 1}$ and $\{\psi_p(n)\}_{n\geqq 1}$ are sub-multiplicative, i.e. for all integers n, k we have*

(a) $$\varphi_q(nk) \leqq \varphi_q(n)\varphi_q(k),$$

(b) $$\psi_p(nk) \leqq \psi_p(n)\psi_p(k).$$

LEMMA 2.2. (a) *If $N > 1$ is an integer, and $q \geqq r \geqq 2$ is defined by $\varphi_r(N) = N^{1/r-1/q}$, then E is of cotype s, for each $s > q$.*

(b) *If $N > 1$ and $p \leqq r \leqq 2$ is defined by $\psi_r(N) = N^{1/p-1/r}$, then E is of type s, for each $s < p$.*

COROLLARY 2.3. *For each integer n, $p_E \leqq p \leqq 2 \leqq q \leqq q_E$,*

$$\varphi_q(n) \geqq n^{(1/q)-(1/q_E)},$$

$$\psi_p(n) \geqq n^{(1/p_E)-(1/p)}.$$

LEMMA 2.4. (a) *If $q \leqq q_E$, then*

$$\lim_{n\to\infty} \frac{\log \varphi_q(n)}{\log n} = \frac{1}{q} - \frac{1}{q_E}.$$

(b) *If $p \geqq p_E$, then*

$$\lim_{n\to\infty} \frac{\log \psi_p(n)}{\log n} = \frac{1}{p_E} - \frac{1}{p}.$$

PROOF. We shall prove (a) only. It follows from the last corollary that

$$\varliminf_{n\to\infty} \frac{\log \varphi_q(n)}{\log n} \geqq \frac{1}{q} - \frac{1}{q_E}.$$

On the other hand, let $\varepsilon > 0$, $r > q_E$, and let n be an integer. By the definition of $\varphi_q(n)$ there exist $y_1, \cdots, y_n$ such that

$$(1-\varepsilon)\varphi_q(n)\left(\int\left\|\sum_{i=1}^n y_i\varepsilon_i(t)\right\|^q dt\right)^{1/q}$$

$$\leqq \left(\sum_{i=1}^n \|y_i\|^q\right)^{1/q}$$

$$\leqq n^{(1/q)-(1/r)}\left(\sum_{i=1}^n \|y_i\|^r\right)^{1/r}$$

$$\leqq C_r n^{(1/q)-(1/r)}\left(\int\left\|\sum_1^n y_i\varepsilon_i(t)\right\|^r dt\right)^{1/r}$$

$$\leqq C_r^{(1)} n^{(1/q)-(1/r)}\left(\int\left\|\sum_1^n y_i\varepsilon_i(t)\right\|^q dt\right)^{1/q}$$

(since E is of cotype r and by Kahane's theorem [2] mentioned above). Hence,

$$\varphi_q(n) \leqq \frac{1}{1-\varepsilon} C_r^{(1)} n^{(1/q)-(1/r)}$$

and therefore

$$\overline{\lim_{n\to\infty}} \frac{\log \varphi_q(n)}{\log n} \leqq \frac{1}{q} - \frac{1}{r}$$

and since $r > q_E$ was arbitrary, we have proved the statement (a). The proof of (b) is precisely the same.

Let us now recall the important theorems of Brunel and Sucheston, and of Krivine, upon which our proof is based.

DEFINITION. Let $\varepsilon > 0$. A sequence $\{y_i\}_{i \leqq k}$ in E is called ε-invariant to spreading, if for all $s \leqq k$, $1 \leqq i_1 < i_2 \cdots < i_s \leqq k$, $1 \leqq j_1 < j_2 \cdots < j_s \leqq k$, and $\alpha_1, \cdots, \alpha_s$ scalars of modulus $\leqq 1$, we have

$$\left\|\sum_{m=1}^s \alpha_m y_{i_m}\right\| \leqq (1+\varepsilon)\left\|\sum_{m=1}^s \alpha_m y_{j_m}\right\|.$$

We shall need the following version of the theorem of Brunel and Sucheston (cf. [1]):

THEOREM 2.5. *Let $\varepsilon > 0$ and k be an integer. Then there exists $N = N(k, \varepsilon)$ such that for every $n \geqq N$ and every sequence $y_1, \cdots, y_n$ in the unit ball of any*

Banach space, there exists a subsequence $y_{i_1}, \cdots, y_{i_k}$ *which is* ε*-invariant to spreading.*

The main deep result on which our proof is based, is the theorem of Krivine [3], in the following finite-dimensional version.

THEOREM 2.6. *Let* $\delta_n \downarrow 0$ *and let* $\{B_n\}_{n \geqq 1}$ *be a sequence of finite-dimensional Banach spaces, such that, for each* n, B_n *is spanned by a sequence* $y_1^n, \cdots, y_n^n$ *of norm-1 elements, which is* δ_n*-invariant to spreading, and unconditional with constant* $\leqq C$, *where* C *does not depend on* n. *If, for every sequence* $n \geqq t_n \to \infty$ *of integers,*

$$p = \inf\left\{ r : \lim_{t_n \to \infty} t_n^{-1/r} \left\| \sum_{i=1}^{t_n} y_i^n \right\| = \infty \right\}$$

then l_p *is block-finitely representable in* $\{B_n\}_{n \geqq 1}$; *i.e., for each* $\varepsilon > 0$ *and each* k, *there exists* n, *and* k *blocks of* $\{y_1^n, \cdots, y_n^n\}$ *which are* $(1+\varepsilon)$*-equivalent to the standard basis of* $l_p^{(k)}$.

Now we proceed to prove the Maurey–Pisier theorem.

THEOREM 2.7. (Maurey and Pisier) *For each infinite-dimensional Banach space* E, l_{p_E} *and* l_{q_E} *are both finitely representable in* E.

PROOF. (a) *For cotype.* Assume that $2 < q_E \leqq \infty$. (If $q_E = 2$, then there is nothing to prove, for, by Dvoretzky's theorem, l_2 is always finitely-representable in E. The case $q_E = \infty$ is simpler, but we prefer to treat it in our general scheme.) It suffices to prove the following assertion:

ASSERTION. There exists $0 < \delta < 1$ such that for each $\varepsilon > 0$ there exists a constant $\gamma > 0$ such that for each integer m there exist m elements $y_1, \cdots, y_m$ of E such that

(i) $1 - \delta \leqq \| y_i \| \leqq 1$, $i = 1, \cdots, m$;

(ii) $(y_1, \cdots, y_m)$ is an unconditional basic sequence with constant $\leqq C$ (C does not depend on m) and is ε-invariant to spreading;

(iii) $\| \Sigma_{j=1}^{s} y_j \| \leqq \gamma \cdot s^{1/q_E}$, for all $s \leqq m$.

Indeed, suppose that the assertion is true. In the case $q_E = \infty$ the inequality (iii) implies $\| \Sigma_1^m a_j \cdot y_j \| \leqq \gamma \max_{1 \leqq j \leqq m} |a_j|$ and therefore $\operatorname{span}\{y_j\}_{j=1}^m$ is $(C \cdot \gamma)$-

isomorphic to l_∞^m. If $q_E < \infty$, then we are in a position to apply Krivine's theorem. For each k, let $y_1^k, \cdots, y_k^k$ be k elements of E satisfying the assertion with $\varepsilon_k = 1/k$, say. Define $z_j^k = y_j^k / \| y_j^k \|$, $j = 1, \cdots, k$. Then, for any $q > q_E$, we shall have, for every $s \leqq k$,

$$s^{1/q} = \left(\sum_{j=1}^{s} \| z_j^k \|^q \right)^{1/q}$$

$$\leqq C_q \left(\int \left\| \sum_{j=1}^{s} z_j^k \varepsilon_j(t) \right\|^q dt \right)^{1/q}$$

$$\leqq C_q \cdot C \left\| \sum_{j=1}^{s} z_j^k \right\| \leqq \frac{\gamma \cdot C}{1-\delta} C_q \cdot s^{1/q_E}.$$

Thus, for any r, and each $s \leqq k$,

$$\frac{s^{(1/q)-(1/r)}}{C \cdot C_q} \leqq s^{-1/r} \left\| \sum_{j=1}^{s} z_j^k \right\| \leqq \frac{\gamma}{1-\delta} s^{(1/q_E)-(1/r)}.$$

Now, for every sequence of integers $k \geqq s_k \to \infty$, it follows that if $r < q_E$ then

$$\lim_k s_k^{-1/r} \left\| \sum_{j=1}^{s_k} z_j^k \right\| = 0$$

and if $r > q_E$, choosing q so that $r > q > q_E$ will yield

$$\lim_k s_k^{-1/r} \left\| \sum_{j=1}^{s_k} z_j^k \right\| = \infty.$$

Hence, l_{q_E} is indeed finitely representable in E, by Theorem 2.6.

Proof of the Assertion. First assume $q_E < \infty$. Let k be given, and let us fix $2 \leqq r < q_E$, so that $k^{(1/r)-(1/q_E)} \leqq 2$. Put $\alpha = \frac{1}{2}(1-(r/q_E))$, and choose n large enough, so that $n^\alpha \geqq N(k, \varepsilon)$, as in Theorem 2.5. In the case $q_E = \infty$ we choose r so large that $k^{1/r} \leqq 2$ and take $\alpha = 1/2$. Then, by the definition of $\varphi_r(n)$, there exist $x_1, \cdots, x_n \in E$ such that

$$\left(\sum_{i=1}^{n} \| x_i \|^r \right)^{1/r} > (1-\varepsilon)\varphi_r(n) \left(\int \left\| \sum_{i=1}^{n} x_i \varepsilon_i(t) \right\|^r dt \right)^{1/r}. \tag{1}$$

With no loss of generality, assume that $\max_i \| x_i \| = 1$. Let $0 < \delta < 1$ and put $w = 1 - \delta$. Define, for each $j \geqq 1$ and $i \geqq 1$,

$$A_j = \{ i : w^j < \| x_i \| \leqq w^{j-1} \},$$

$$\hat{x}_i = x_i / w^{j-1}, \qquad \text{for } i \in A_j.$$

If $|A_j| \leqq n^\alpha$, we set $A_{j,0} = A_j$ and do nothing else with this set. If $|A_j| > n^\alpha$, then, by Theorem 2.5 and the choice of n, there exists a subset $A_{j,1}^{(1)} \subset A_j$ of k elements which are ε-invariant to spreading. We choose a subset $A_{j,1} \subset A_{j,1}^{(1)}$ for which

$$\left(\int \left\| \sum_{i \in A_{j,1}} \hat{x}_i \varepsilon_i(t) \right\|^r dt\right)^{1/r} \Big/ |A_{j,1}|^{1/q_E}$$

is maximal. If $|A_j - A_{j,1}| > n^\alpha$, then there exist two other subsets $A_{j,2} \subset A_{j,2}^{(1)} \subset A_j - A_{j,1}$ with the same properties. Continuing in this manner we obtain a disjoint partition

$$A_j = A_{j,1} \cup \cdots \cup A_{j,m_j} \cup A_{j,0},$$

where $|A_{j,0}| \leqq n^\alpha$, and for each $1 \leqq p \leqq m_j$, $A_{j,p} \subset A_{j,p}^{(1)}$, where $A_{j,p}^{(1)}$ consists of k elements which are ε-invariants to spreading, and for each $T \subset A_{j,p}^{(1)}$,

$$\frac{\left(\int \left\| \sum_{i \in A_{j,p}} \hat{x}_i \varepsilon_i(t) \right\|^r dt\right)^{1/r}}{|A_{j,p}|^{1/q_E}} \geqq \frac{\left(\int \left\| \sum_{i \in T} \hat{x}_i \varepsilon_i(t) \right\|^r dt\right)^{1/r}}{|T|^{1/q_E}}.$$

We now split $\{1, \cdots, n\}$ into two parts. The "bad" part is $A' = \bigcup_j A_{j,0}$ and the "good" part, its complement, is $A'' = \bigcup_j \bigcup_{1 \leqq t \leqq m_j} A_{j,t}$. Let us show first that the "bad" part can be essentially ignored in inequality (1). Indeed,

$$\begin{aligned}\sum_{i=1}^n \|x_i\|^r &= \sum_{i \in A'} \|x_i\|^r + \sum_{i \in A''} \|x_i\|^r \\ &\leqq \sum_j w^{(j-1)r} |A_{j,0}| + \sum_{i \in A''} \|x_i\|^r \\ &\leqq \frac{n^\alpha}{1 - w^r} + \sum_{i \in A''} \|x_i\|^r.\end{aligned}$$

But α is so chosen that, by the property of $\varphi_r(n)$ in Corollary 2.3, $n^\alpha = o(\varphi_r(n)^r)$, and since $1 \leqq \int \|\sum_{i=1}^n x_i \varepsilon_i(t)\|^r dt$ (because $\max \|x_i\| = 1$, and the symmetry properties of the Rademacher variables), we may assume that

$$\frac{n^\alpha}{1 - w^r} < \varepsilon(1-\varepsilon)^r \varphi_r(n)^r \cdot \int \left\| \sum_{i=1}^n x_i \varepsilon_i(t) \right\|^r dt.$$

Thus, it follows from (1) that

$$(1-\varepsilon)^{r+1} \varphi_r(n)^r \int \left\| \sum_{i=1}^n x_i \varepsilon_i(t) \right\|^r dt \leqq \sum_{i \in A''} \|x_i\|^r$$

and by the symmetry properties of the Rademacher variables,

$$(1-\varepsilon)^{r+1}\varphi_r(n)^r\int\Big\|\sum_{i\in A''}x_i\varepsilon_i(t)\Big\|^r dt \leqq \sum_{i\in A''}\|x_i\|^r$$

so that we have a similar inequality to (1), involving only the "good" part of $\{1,\cdots,n\}$.

Let us now introduce new symbols $\{B_s\}_{s=1}^l$ for the subsets $\{A_{j,t}\}_{j,1\leqq t\leqq m_j}$. Let us define

$$u_s(\theta)=\sum_{i\in B_s}x_i\varepsilon_i(\theta),\qquad s\leqq l,\quad \theta\in[0,1].$$

Then

$$\sum_{s=1}^{l}\|u_s(\theta)\|^r\leqq\varphi_r(l)^r\int\Big\|\sum_{s=1}^{l}\sum_{i\in B_s}x_i\varepsilon_i(\theta)\varepsilon_s(t)\Big\|^r dt.$$

Integrating with respect to θ, we obtain

$$\begin{aligned}\sum_{s=1}^{l}\int\|u_s(\theta)\|^r d\theta &\leqq \varphi_r(l)^r\int\Big\|\sum_{i\in A''}x_i\varepsilon_i(\theta)\Big\|^r d\theta\\ &\leqq\frac{\varphi_r(l)^r}{(1-\varepsilon)^{r+1}\varphi_r(n)^r}\sum_{i\in A''}\|x_i\|^r\\ &=\frac{\varphi_r(l)^r}{(1-\varepsilon)^{r+1}\varphi_r(n)^r}\sum_{s=1}^{l}\Big(\sum_{i\in B_s}\|x_i\|^r\Big).\end{aligned}$$

Hence, there must exist at least one index $1\leqq s_0\leqq l$ such that

$$\begin{aligned}\int\|u_{s_0}(\theta)\|^r d\theta &= \int\Big\|\sum_{i\in B_{s_0}}x_i\varepsilon_i(\theta)\Big\|^r d\theta\\ &\leqq\frac{\varphi_r(l)^r}{(1-\varepsilon)^{r+1}\varphi_r(n)^r}\sum_{i\in B_{s_0}}\|x_i\|^r.\end{aligned}$$

By replacing the x_i's by the normalized $\hat{x}_i$'s and by making ε small enough, we have (observing that $\varphi_r(l)\leqq\varphi_r(n)$)

$$\Big(\int\Big\|\sum_{i\in B_{s_0}}\hat{x}_i\varepsilon_i(\theta)\Big\|^r d\theta\Big)^{1/r}\leqq 2\Big(\sum_{i\in B_{s_0}}\|\hat{x}_i\|^r\Big)^{1/r}\leqq 2|B_{s_0}|^{1/r}.$$

But $|B_{s_0}|\leqq k$, and by our choice of the number r, we obtain

$$\Big(\int\Big\|\sum_{i\in B_{s_0}}\hat{x}_i\varepsilon_i(\theta)\Big\|^r d\theta\Big)^{1/r}\leqq 4|B_{s_0}|^{1/q_E}$$

(in the case $q_E=\infty$ we have formally the same inequality because $|B_{s_0}|^{1/r}\leqq k^{1/r}\leqq$

2). Since the set B_{s_0} is one of the sets $A_{j,t}$ it is contained in some set $A \equiv A^{(1)}_{j_0,t_0}$ which consists of k elements that are ε-invariant to spreading. By the choice of B_{s_0}, we obtain, for each $T \subset A$,

$$\left(\int \left\| \sum_{i \in T} \hat{x}_i \varepsilon_i(\theta) \right\|^r d\theta \right)^{1/r} \leqq 4 |T|^{1/q_E}. \tag{2}$$

Thus, to complete the proof of the assertion, it remains to show the existence of $T \subset A$, which is large enough and forms a "good" unconditional basic sequence. To do so, we make use of the following lemma, due to Brunel and Sucheston [1]:

LEMMA 2.8. *Let $\delta, \varepsilon > 0$. For each n there exists $N = N(n, \delta, \varepsilon)$ such that for each N elements $x_1, \cdots, x_N$ in the unit ball of any Banach space, satisfying $\|x_i - x_j\| \geqq \delta$ for each $i \neq j$, there exists a subsequence $x_{i_1}, x_{i_2}, \cdots, x_{i_{2n}}$ such that the sequence $v_j \equiv x_{i_{2j}} - x_{i_{2j-1}}$, $j = 1, \cdots, n$, is an unconditional basic sequence with constant $\leqq 2 + \varepsilon$, and for each $j \leqq n$, $\|v_j\| \geqq \delta$.*

Let $T_0 \subset A$ be a subset such that $\{\hat{x}_i\}_{i \in T_0}$ is contained in a ball of radius δ. Since all of these elements have norm $\geqq 1 - \delta$, it follows that there exists a functional $x^* \in E^*$ such that $\|x^*\| = 1$ and $|x^*(\hat{x}_i)| \geqq 1 - 3\delta$ for each $i \in T_0$. Then, from (2) we obtain

$$\left(\int \left| \sum_{i \in T_0} x^*(\hat{x}_i) \varepsilon_i(\theta) \right|^r d\theta \right)^{1/r} \leqq \left(\int \left\| \sum_{i \in T_0} \hat{x}_i \varepsilon_i(\theta) \right\|^r d\theta \right)^{1/r} \leqq 4 |T_0|^{1/q_E}.$$

But it is well-known that for any scalars $\{\alpha_i\}$

$$(r \geqq 2) \qquad \left(\int \left| \sum_i \alpha_i \varepsilon_i(\theta) \right|^r d\theta \right)^{1/r} \geqq \left(\sum_i |\alpha_i|^2 \right)^{1/2}.$$

Hence, $(1 - 3\delta)|T_0|^{\frac{1}{2}} \leqq 4 |T_0|^{1/q_E}$ and so

$$|T_0| \leqq \left(\frac{4}{1 - 3\delta} \right)^{1/((1/2) - (1/q_E))} \equiv M.$$

To conclude, no more than a fixed number M of elements of $\{\hat{x}_i\}_{i \in A}$ can be contained in a ball of radius δ. Hence, by a standard argument, there is a subset $T \subset A$ of at least k/M points such that for any $i \neq j \in T$, $\|\hat{x}_i - \hat{x}_j\| \geqq \delta$. Hence, by Lemma 2.8, if k is large enough, we can find a subsequence $\hat{x}_{i_1}, \cdots, \hat{x}_{i_{2m}}$ such that the sequence $v_j \equiv \hat{x}_{i_{2j}} - \hat{x}_{i_{2j-1}}$, $j = 1, \cdots, m$ is unconditional with a constant $\leqq 3$, say, and $\|v_j\| \geqq \delta$ for all $j \leqq m$. But from (2) we obtain, for each $s \leqq m$,

$$\left\|\sum_{j=1}^{s} v_j\right\| \leqq 3\left(\int\left\|\sum_{j=1}^{s} v_j\varepsilon_j(\theta)\right\|^r d\theta\right)^{1/r}$$

$$\leqq 3\left(\int\left\|\sum_{j=1}^{s} \hat{x}_{i_{2j}}\varepsilon_j(\theta)\right\|^r d\theta\right)^{1/r} + 3\left(\int\left\|\sum_{j=1}^{s} \hat{x}_{i_{2j-1}}\varepsilon_j(\theta)\right\|^r d\theta\right)^{1/r}$$

$$\leqq 24 s^{1/q_E}.$$

Whence the assertion follows, by setting $y_j = v_j/2$, $j \leqq m$, with $1-\delta/2$ instead of δ, and $\gamma = 12$.

(b) *For type.* The proof is quite similar to the previous one, with some differences, on which we shall now comment. As above, we may assume that $p_E < 2$, for otherwise there is nothing to prove. Instead of (iii) in the assertion, we require that for all $s \leqq m$

(iii′) $\|\sum_{j=1}^{s} y_j\| \geqq (1/\gamma(s))s^{1/p_E}$

where $\gamma(s) = K(\psi_{p_E}(s))^\beta$ for some constants $K > 0$ and $\beta \geqq 0$ that will be chosen later in the proof. For any such choice, we have by Lemma 2.4

$$\lim_{s\to\infty}\frac{\log \gamma(s)}{\log s} = 0.$$

Hence, the application of Krivine's theorem can be carried out in much the same way as in the proof for cotype (except for $p_E = 1$, in which case we choose $p = p_E$ for all k, and since any Banach space is of type 1, $C_p = C_1 < \infty$, and the proof becomes simpler).

In the proof of the assertion, we fix $k > 1$ and choose $p_E < r < 2$ such that $k^{(1/p_E)-(1/r)} < 2$, and $\alpha = \frac{1}{2}((r/p_E)-1)$, using ψ instead of φ, we obtain the following derivation (which, up to formula (4), is rather similar to the derivation of (2) above, but is given here to reflect some minor differences):

$$\left(\int\left\|\sum_{i=1}^{n} x_i\varepsilon_i(t)\right\|^r dt\right)^{1/r} > (1-\varepsilon)\psi_r(n)\left(\sum_{i=1}^{n}\|x_i\|^r\right)^{1/r}; \tag{3}$$

w, A_j, $\hat{x}_i$, $A_{j,t}^{(1)}$ and $A_{j,t}$ are defined precisely as before (but now we require that $A_{j,t} \subset A_{j,t}^{(1)}$ should satisfy

$$\frac{\left(\int\left\|\sum_{i\in A_{j,t}}\hat{x}_i\varepsilon_i(t)\right\|^r dt\right)^{1/r}}{|A_{j,t}|^{1/p_E}} \leqq \frac{\left(\int\left\|\sum_{i\in T}\hat{x}_i\varepsilon_i(t)\right\|^r dt\right)^{1/r}}{|T|^{1/p_E}}$$

for all $T \subset A_{j,t}^{(1)}$. Now we construct, in a similar way as before, a partition of $\{1,\cdots,n\}$ into a "bad" part $A' = \bigcup_j A_{j,0}$ and a "good" part A'', its complement.

Again, the "bad" part A' can be ignored in (3), as follows:

By the triangle inequality,

$$\left(\int\left\|\sum_{i=1}^{n} x_i\varepsilon_i(t)\right\|^r dt\right)^{1/r} \leqq \left(\int\left\|\sum_{i\in A''} x_i\varepsilon_i(t)\right\|^r dt\right)^{1/r} + \sum_{i\in A'} \|x_i\|.$$

But

$$\sum_{i\in A'} \|x_i\| \leqq \sum_j w^{(j-1)}|A_{j,0}| \leqq \frac{n^\alpha}{1-w} \leqq \frac{n^\alpha}{1-w}\left(\sum_{i=1}^{n} \|x_i\|^r\right)^{1/r}$$

and, since $r<2$, one has, from Corollary 2.3, $n^\alpha = o(\psi_r(n))$. This, and the obvious inequality

$$\left(\sum_{i=1}^{n} \|x_i\|^r\right)^{1/r} \geqq \left(\sum_{i\in A''} \|x_i\|^r\right)^{1/r},$$

yield that if n is large enough, then by (3)

$$\left(\int\left\|\sum_{i\in A''} x_i\varepsilon_i(t)\right\|^r dt\right)^{1/r} > (1-2\varepsilon)\psi_r(n)\left(\sum_{i\in A''} \|x_i\|^r\right)^{1/r}.$$

$\{B_s\}_{s=1}^{l}$ and $\{u_s(\theta)\}_{s=1}^{l}$ are defined precisely as above, and, for each $\theta\in[0,1]$,

$$\int\left\|\sum_{s=1}^{l}\sum_{i\in B_s} x_i\varepsilon_i(\theta)\varepsilon_s(t)\right\|^r dt \leqq \psi_r(l)^r\sum_{s=1}^{l}\|u_s(\theta)\|^r$$

and by integrating with respect to θ,

$$\int\left\|\sum_{i\in A''} x_i\varepsilon_i(\theta)\right\|^r d\theta \leqq \psi_r(l)^r\sum_{s=1}^{l}\int\|u_s(\theta)\|^r d\theta$$

and therefore

$$\sum_{s=1}^{l}\int\|u_s(\theta)\|^r d\theta > \frac{(1-2\varepsilon)^r\psi_r(n)^r}{\psi_r(l)^r}\sum_{s=1}^{l}\left(\sum_{i\in B_s}\|x_i\|^r\right).$$

Hence, there must exist at least one index $1\leqq s_0\leqq l$ such that

$$\int\|u_{s_0}(\theta)\|^r d\theta > \frac{(1-2\varepsilon)^r\psi_r(n)^r}{\psi_r(l)^r}\sum_{i\in B_{s_0}}\|x_i\|^r$$

and, as before, one can obtain

$$\left(\int\left\|\sum_{i\in B_{s_0}} \hat{x}_i\varepsilon_i(\theta)\right\|^r d\theta\right)^{1/r} > \tfrac{1}{2}|B_{s_0}|^{1/r}.$$

Again, by the choice of r and $B_{s_0}\equiv A_{j_0,t_0}$, we obtain, for each $T\subset A\equiv A_{j_0,t_0}^{(1)}$

$$\left(\int\left\|\sum_{i\in T} \hat{x}_i\varepsilon_i(\theta)\right\|^r d\theta\right)^{1/r} > \tfrac{1}{4}|T|^{1/p_E}. \tag{4}$$

As in the proof for cotype, we want now to make use of Lemma 2.8 in order to obtain an unconditional basic sequence that satisfies our assertion. However, here we face extra difficulties due to the direction of the inequality (4), which prevents us from applying the triangle inequality in a simple manner, as we did in the case of cotype.

We proceed as follows: First, we claim that there exists $\delta > 0$ such that for each integer l we can find k large enough so that the set $\{\hat{x}_i\}_{i=1}^k$ which satisfies (4) contains a subset $\{u_i\}_{i=1}^{2l}$ such that

$$\|u_i - u_j\| \geqq \delta, \qquad 1 \leqq i \neq j \leqq 2l. \tag{5}$$

Indeed, fix m such that $\frac{1}{4} m^{1/p_E} > 1 + m^{\frac{1}{2}}$ and let $\delta = 1/m$. Take $k > 2lm$. It is sufficient to show that no ball of radius δ can contain m points of the sequence $\{\hat{x}_i\}_{i=1}^k$ satisfying (4). Thus let x_0 be a center of such a ball which contains, say, $\hat{x}_1, \cdots, \hat{x}_m$. By (4) we get

$$\begin{aligned}
\tfrac{1}{4} m^{1/p_E} &< \left(\int \left\| \sum_{i=1}^m \hat{x}_i \varepsilon_i(\theta) \right\|^r d\theta \right)^{1/r} \\
&\leqq \left(\int \left\| \sum_{i=1}^m (\hat{x}_i - x_0) \varepsilon_i(\theta) \right\|^r d\theta \right)^{1/r} + \left(\int \left\| \sum_{i=1}^m x_0 \varepsilon_i(\theta) \right\|^r d\theta \right)^{1/r} \\
&\leqq \sum_{i=1}^m \|\hat{x}_i - x_0\| + m^{\frac{1}{2}} \\
&\leqq 1 + m^{\frac{1}{2}}
\end{aligned}$$

contradicting the choice of m.

Thus, there exists $\delta > 0$ such that for each integer l we can find a sequence of almost norm 1 elements $\{u_i\}_{i=1}^{2l} \subset E$ which is ε-invariant to spreading, satisfies (5) and

$$\left(\int \left\| \sum_{i \in T} u_i \varepsilon_i(\theta) \right\|^r d\theta \right)^{1/r} > \tfrac{1}{4} |T|^{1/p_E}, \qquad \text{for all } T \subset \{1, \cdots, 2l\}. \tag{6}$$

By Lemma 2.8 we may also assume that the sequence $\{y_j = u_{2j-1} - u_{2j}\}_{j=1}^l$ is an unconditional basic sequence with an unconditional constant $\leqq 3$, is ε-invariant to spreading, and $\delta \leqq \|y_j\| \leqq 2$ for all $j \leqq l$. Thus $\{\frac{1}{2} y_j\}_{j=1}^l$ satisfies conditions (i) and (ii) of the assertion, and the following lemma asserts that a subsequence of this sequence satisfies condition (iii′).

LEMMA 2.9. *There exist constants $K, K_0 > 0$, $\beta \geqq 0$, such that for each integer m, if we take*

$$l > m\left(\left[(K_0\psi_{p_E}(m))^{2p_E/(2-p_E)}\right]+1\right)$$

and construct a sequence $\{y_j\}_{j=1}^{l}$ *as above, then* $\{\frac{1}{2}y_j\}_{j=1}^{m}$ *satisfies condition* (iii′) *of the assertion, i.e.*

$$\left\|\sum_{j=1}^{s}\tfrac{1}{2}y_j\right\| \geqq \frac{1}{\gamma(s)}\cdot s^{1/p_E}, \qquad s = 1,\cdots,m$$

where $\gamma(s) = K(\psi_{p_E}(s))^{\beta}$.

PROOF. Let $C > 0$ be a constant independent of r such that the $L^{p_E}(E)$ and $L^r(E)$ norms are C-equivalent on the span of elements of the form $x\cdot\varepsilon_i(\theta)$, $x \in E$. (C can be chosen to be independent of r since r belongs to the bounded interval $[p_E, 2]$.) Put $K_0 = 9C$. Suppose that there exists $s \leqq m$ such that

$$\left\|\sum_{j=1}^{s} y_j\right\| < \frac{2}{\gamma(s)} s^{1/p_E}. \tag{7}$$

Let

$$t = [(K_0\psi_{p_E}(s))^{2p_E/(2-p_E)}].$$

By the choice of l we have $s(t+1) \leqq l$, so that we can construct the following $t \times s$ matrix $U = (U_{ij})$ from the set $\{u_j\}_{j=1}^{2l}$:

$$\begin{array}{llll}
u_1 - u_2 & u_{t+2} - u_{t+3} & \cdots & u_{(t+1)(s-1)+1} - u_{(t+1)(s-1)+2} \\
u_1 - u_3 & u_{t+2} - u_{t+4} & \cdots & \\
 & & \cdots & \vdots \\
u_1 - u_{t+1} & u_{t+2} - u_{2(t+1)} & \cdots & u_{(t+1)(s-1)+1} - u_{s(t+1)}.
\end{array}$$

Let $\{\alpha_{ij}(\theta)\}_{i\leqq t, j\leqq s}$ be an enumeration of independent Rademacher functions, and define

$$X(\theta) = \sum_{i=1}^{t}\sum_{j=1}^{s}\alpha_{ij}(\theta)U_{ij}.$$

Then we have

$$\left(\int\|X(\theta)\|^r d\theta\right)^{1/r} \leqq \sum_{i=1}^{t}\left(\int\left\|\sum_{j=1}^{s}\alpha_{ij}(\theta)U_{ij}\right\|^r d\theta\right)^{1/r}.$$

But by the ε-invariance to spreading and the unconditionality of the u_j's, we have for each θ

$$\left\|\sum_{j=1}^{s}\alpha_{ij}(\theta)U_{ij}\right\| \leqq (1+\varepsilon)\left\|\sum_{j=1}^{s}\alpha_{ij}(\theta)(u_{2j-1}-u_{2j})\right\|$$

$$= (1+\varepsilon)\left\| \sum_{j=1}^{s} \alpha_{ij}(\theta) y_j \right\|$$

$$\leqq 3(1+\varepsilon)\left\| \sum_{j=1}^{s} y_i \right\|$$

$$< \frac{6(1+\varepsilon)}{\gamma(s)} s^{1/p_E}$$

by (7). Hence

$$\left(\int \| X(\theta) \|^r d\theta \right)^{1/r} < \frac{6(1+\varepsilon)t}{\gamma(s)} \cdot s^{1/p_E}.$$

On the other hand, by the triangle inequality,

$$\left(\int \| X(\theta) \|^r d\theta \right)^{1/r} \geqq \left(\int \left\| \sum_{i=1}^{t} \sum_{j=1}^{s} u_{(t+1)(j-1)+i+1} \alpha_{ij}(\theta) \right\|^r d\theta \right)^{1/r}$$

$$- \left(\int \left\| \sum_{i=1}^{t} \sum_{j=1}^{s} u_{(t+1)(j-1)+1} \alpha_{ij}(\theta) \right\|^r d\theta \right)^{1/r}$$

$$= I_1 - I_2.$$

But I_1 can be written as $(\int \| \Sigma_{i \in T} u_i \varepsilon_i(\theta) \|^r d\theta)^{1/r}$, where $T = \{2, 3, \cdots, t+1, t+3, \cdots, 2(t+1), 2(t+1)+2, \cdots, s(t+1)\}$. Hence, by (6),

$$| I_1 | \geqq \tfrac{1}{4} | T |^{1/p_E} = \tfrac{1}{4} t^{1/p_E} \cdot s^{1/p_E}.$$

As for the second integral, we have

$$I_2 = \left(\int \left\| \sum_{j=1}^{s} \left(\sum_{i=1}^{t} \alpha_{ij}(\theta) \right) u_{(t+1)(j-1)+1} \right\|^r d\theta \right)^{1/r}.$$

Define $h_j(\theta) = \Sigma_{i=1}^{t} \alpha_{ij}(\theta)$, $j = 1, \cdots, s$. These functions are symmetric and independent, and therefore, by the symmetric properties of the Rademacher functions, we have

$$I_2 = \left(\int \int \left\| \sum_{j=1}^{s} h_j(\theta) \varepsilon_j(\xi) u_{(t+1)(j-1)+1} \right\|^r d\theta d\xi \right)^{1/r}.$$

Integrating first with respect to ξ and using the definition of $\psi_{p_E}(s)$, we obtain

$$I_2 \leqq C \cdot \psi_{p_E}(s) \left(\sum_{j=1}^{s} \int | h_j(\theta) |^{p_E} d\theta \right)^{1/p_E}$$

$$\leqq C \cdot \psi_{p_E}(s) \left(\sum_{j=1}^{s} \left(\int | h_j(\theta) |^2 d\theta \right)^{p_E/2} \right)^{1/p_E}$$

$$= C \cdot t^{\frac{1}{2}} \cdot \psi_{p_E}(s) \cdot s^{1/p_E}.$$

Combining all inequalities together, we finally obtain

$$\frac{6(1+\varepsilon)t}{\gamma(s)}\cdot s^{1/p_E} > \tfrac{1}{4}t^{1/p_E}\cdot s^{1/p_E} - Ct^{\frac{1}{2}}\psi_{p_E}(s)\cdot s^{1/p_E}$$

or, by the choice of t,

$$\frac{6(1+\varepsilon)t^{\frac{1}{2}}}{\gamma(s)} > \tfrac{1}{4}t^{1/p_E-\frac{1}{2}} - C\cdot\psi_{p_E}(s) \geqq C\psi_{p_E}(s).$$

Hence,

$$\gamma(s) < \frac{6(1+\varepsilon)(9C\psi_{p_E}(s))^{p_E/(2-p_E)}}{C\cdot\psi_{p_E}(s)}$$

$$= 54(1+\varepsilon)\cdot(9C)^{(p_E-1)/(1-p_E/2)}\cdot(\psi_{p_E}(s))^{(p_E-1)/(1-p_E/2)}.$$

Thus, if we choose $\beta = (p_E-1)/(1-p_E/2) \geqq 0$, $K = 54(1+\varepsilon)(9C)^\beta$, we will obtain a contradiction to the definition of $\gamma(s)$. This proves the lemma, and the proof of the theorem is thus completed. Q.E.D.

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